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LETTER TO THE EDITOR

Large coupling behaviour of the Lyaponov exponent for tight binding one-dimensional random systems†

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Abstract. We study the Lyaponov exponent $\gamma_\lambda(E)$ of $(hu)(n) = u(n+1) + u(n-1) + \lambda V(n)u(n)$ in the limit as $\lambda \to \infty$ where V is a suitable random potential. We prove that $\gamma_\lambda(E) \sim \ln \lambda$ as $\lambda \to \infty$ uniformly as E/λ runs through compact sets. We also describe a formal expansion (to order λ^{-2}) for random and almost periodic potentials.

In this note, we study one-dimensional tight binding Hamiltonians $h = h_0 + \lambda V$ where

$$(h_0 u)(n) = u(n+1) + u(n-1). \tag{1}$$

We are interested in the cases where V is either random or almost periodic. By random, we mean that $V(n)$ is a family of identically distributed independent random variables with density $P(y)\,\mathrm{d}y$ where P is bounded with bounded support. In the random case, we will succeed in identifying the first few terms in the large λ behaviour of the Lyaponov exponent. For the almost periodic case only a formal large λ expansion is obtained.

Explicitly, we let $\gamma_\lambda(E)$ be the Lyaponov exponent for h, i.e.

$$\gamma_\lambda(E) = \lim_{n\to\infty} \frac{1}{n} \ln \|M_n(\omega) \ldots M_1(\omega)\|$$

where

$$M_j(\omega) = \begin{pmatrix} E - \lambda V_\omega(j) & -1 \\ 1 & 0 \end{pmatrix}.$$

The limit exists for typical ω.

We will prove that in the random case as $\lambda \to \infty$, for E fixed,

$$\gamma_\lambda(\lambda E) - \ln \lambda - \int \ln|E - E'| P(E')\,\mathrm{d}E' \to 0. \tag{2}$$

As we will explain, we believe but will not prove that the left side of (2) is actually $\mathrm{O}(\lambda^{-2})$. A formal calculation of the λ^{-2} term will be given subsequently.

Our proof of (2) begins with the Thouless formula (see Herbert and Jones (1971), Thouless (1972) for the original arguments, Avron and Simon (1983), Craig and

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Simon (1983) for rigorous proofs):

$$\gamma_\lambda(E) = \int \ln|E - E'| \, dk_\lambda(E') \tag{3}$$

where k_λ is the integrated density of states for h. Now, define

$$\tilde{k}_\lambda(E) = k_\lambda(\lambda E). \tag{4}$$

Changing variables in (3), we get

$$\gamma_\lambda(\lambda E) = \ln \lambda + \int \ln|E - E'| \, d\tilde{k}_\lambda(E'). \tag{5}$$

The point of this change of variables is that while k_λ does not have a limit as $\lambda \to \infty$, $\tilde{k}_\lambda$, as the density of states of $\lambda^{-1}h_0 + V$, should have a decent limit. Indeed, if $k_\infty(E) = \int_{-\infty}^{E} P(y)\, dy$, then since $\|h_0\| = 2$, we have

$$k_\infty(E - 2\lambda^{-1}) \leq \tilde{k}_\lambda(E) \leq k_\infty(E + 2\lambda^{-1}) \tag{6}$$

from which one easily sees that

$$\int f(y)\, d\tilde{k}_\lambda(y) \to \int f(y) P(y)\, dy \tag{7}$$

for any bounded continuous f. Given (5), we have proven (2) if we can control the error made by replacing $\ln|y - E'|$ by a continuous function cut-off very near $y = E$ (so that (7) holds). Such an error is easy to control if one has that

$$|\tilde{k}_\lambda(E) - \tilde{k}_\lambda(E')| \leq C|E - E'|^\alpha \tag{8}$$

for some $C, \alpha > 0$ fixed uniformly in all large λ. For the random case under discussion, (8) with $\alpha = 1$ is a result of Wegner (1983).

This result of Wegner completes the proof of (2) in the random case. If one makes the estimates in (6) and (8) explicit, one finds that we have shown that the left side of (2) goes to zero at least as fast as $\lambda^{-1/2}$. Actually, we expect that the difference is $O(\lambda^{-2})$. We shall later describe how to derive a formal expansion for large λ and explicitly evaluate the λ^{-2} terms both for the almost periodic (AP) and random cases. A *small* λ formal expansion has been described by Derrida (1982).

The $\log \lambda$ growth also agrees with the known growth of γ in the two cases in which it is known that γ is positive for almost periodic V: the Aubry-André argument for the almost Mathieu equation (André and Aubry 1980, Avron and Simon 1982, 1983) and Herman's argument (1982) that $\gamma \geq \ln(\lambda/2)$ if $V(x) = \cos(2\pi\beta n) + \sum_{j=0}^{n-1} a_j \cos(2\pi\beta j)$ with β irrational. It seems to us quite likely that in many AP cases (8) holds for sufficiently small $\alpha \leq \frac{1}{2}$ and sufficiently large λ, but the proof eludes us. In particular, if $V(n) = f(2\pi\beta n)$, where f is a smooth function on the circle with only non-degenerate critical points, the limiting distribution k_∞ obeys (8), so k_λ should as well. Such a result, which would imply that $\gamma_\lambda(E) > 0$ for all E when λ is sufficiently large, would be interesting because of the consequences of a positive Lyaponov exponent.

We make three remarks. Our proof that $\gamma > 0$ for λ large avoided Furstenberg's theorem. One can understand where $\ln \lambda$ comes from: if M is replaced by

$$\begin{pmatrix} \lambda(E - \nu) & 0 \\ 0 & 0 \end{pmatrix},$$

then obviously γ grows like $\ln \lambda$. Finally, in the periodic case, there is an obvious version of equation (2) which holds for energies away from the finite number of values of the λV. Here uniformity is clearly lost.

We close with a short description of a formal large λ expansion. In the case of random potential we find, to order λ^{-2},

$$\gamma(\lambda E) = \ln \lambda + \int P(E') \ln|E'-E| \,\mathrm{d}E - \frac{1}{\lambda^2} \operatorname{Re}\left[\left(\int \frac{P(E')}{E'-E-\mathrm{i}0} \mathrm{d}E'\right)^2\right] + \ldots \tag{9}$$

For an AP potential $V_n = f(\beta n)$ with β irrational and f periodic of period one we find

$$\gamma(\lambda E) = \ln \lambda + \int_0^1 \mathrm{d}x \ln|f(x)-E| - \frac{1}{\lambda^2} \operatorname{Re} \int_0^1 \frac{\mathrm{d}x}{(f(x)-E-\mathrm{i}0)(f(x+\omega)-E-\mathrm{i}0)} + \ldots \tag{10}$$

Equation (10) has an obvious generalisation to more complicated AP potentials. It is interesting that while the first two terms are identical in (9) and (10), the λ^{-2} term is different.

An easy derivation of (9) and (10) follows from the random walk expansion for the resolvent (perturbation expansion in h_0):

$$\left(\frac{1}{h-E}\right)(m, n) = -\sum_{N=0} \sum_{\substack{\omega_j \omega_0 = m \\ \omega_N = n}} \prod_{j=0} \left(\frac{-1}{\lambda V(\omega_j) - E}\right), \tag{11}$$

the sum being over all paths in N steps from n to m. Recall that the density of states

$$\mathrm{d}k/\mathrm{d}E = \pi^{-1} \operatorname{Im}\langle (h-E-\mathrm{i}0)^{-1}(n, n)\rangle \tag{12}$$

(where $\langle\cdot\rangle$ means expectation over the class of potentials) and that the integrated density of states and the Lyaponov exponents are the real and imaginary parts of an analytic function in the upper half plane.

Now integrate $(h-E)^{-1}(n, n)$ with respect to E term by term. The $N=0$ term gives the first two terms in (9) and (10). There is no $N=1$ term and for the $N=2$ term use

$$\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{1}{(y-x)(x-E)} - \frac{1}{(y-x)^2} \ln \frac{y-E}{x-E}\right) = \frac{1}{(x-E)^2(y-E)}.$$

Translation invariance and some algebra give the third terms. Higher powers in λ^{-1} involve integrating more complicated rational functions of E.

One can convince oneself by looking at the case $f = 2\cos(2\pi x)$ that the third term in (10) is not uniformly bounded in energy.

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